Estimating the Probability Distribution of von Mises Stress for Structures Undergoing Random Excitation, Part 2: Example Calculations

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Abstract

The primary purpose of finite element stress analysis is to estimate the reliability of engineering designs. In structural applications, the von Mises stress due to a given load is often used as the metric for evaluating design margins. For deterministic loads, both static and dynamic, the calculation of von Mises stress is straightforward [2]. For random load environments typically defined in terms of power spectral densities, however, the deterministic theory normally applied to compute RMS acceleration, displacement, or stress tensor responses cannot be applied directly to calculate the probability distribution of von Mises stress, a nonlinear function of the linear stress components. Recently methods have been developed to solve this problem [4]. Those methods and are applied here to an example structure.

1. Introduction

Failure of ductile structures is associated with von Mises stresses that exceed the strength of the material. A first level reliability survey would consist of calculating RMS values of von Mises stress and assuring that those values are comfortably less than the strength of the material involved[3]. The next level of scrutiny would involve computing the probability distribution of von Mises stress at regions of high RMS value, to assess what fraction of the time that material exceeds some threshold value of von Mises stress.

In an accompanying paper [4], Segalman and Reese derive the probability integral for von Mises stress resulting from the application of Gaussian, zero-mean loads to a linear structure. They also show how that integral can be approximated conveniently. Here, we illustrate the application of those methods to a specific structure.

Because the von Mises stress is a nonlinear, (non-negative) function of the stress tensor, the probability distribution of von Mises stress cannot be Gaussian, even if the components of the stress tensor each have Gaussian distribution. In general, the form of the von Mises probability distribution is not known *a priori*. Design of structures to achieve a desired level of reliability with respect to von Mises stress requires calculation of both the form of the probability distribution and the parameters of that distribution.

The reliability of structures of ductile material is assessed in terms of the probability distribution of the von Mises stress. That distribution depends on both the structure and the loading applied, so at minimum, calculation of the distribution requires a linear model for the structure and a statistical specification of the input forces. In principle, from the linear model one can deduce all required transfer functions. The input forces are typically specified by their auto spectral densities with all phase content missing. In the case of multiple force inputs, the forces may be specified by a cross spectral density matrix over frequency. The method illustrated here uses exactly that information, along with a linear model for the structure, to calculate the probability distribution of von Mises stress.

2. Analytical reliability model

The von Mises stress is a function of the stress tensor:

$$p^2(t, x) = \sigma(t, x)^T A\, \sigma(t, x), \qquad \text{Eq 1}$$

where

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$$A = \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} & & & \\ -\frac{1}{2} & 1 & -\frac{1}{2} & & & \\ -\frac{1}{2} & -\frac{1}{2} & 1 & & & \\ & & & 3 & & \\ & & & & 3 & \\ & & & & & 3 \end{bmatrix} \text{ and } \sigma(t, x) = \begin{bmatrix} \sigma_{11}(t, x) \\ \sigma_{22}(t, x) \\ \sigma_{33}(t, x) \\ \tau_{12}(t, x) \\ \tau_{13}(t, x) \\ \tau_{23}(t, x) \end{bmatrix} \qquad \text{Eq 2}$$

Here, $\sigma(t, x)$ is the stress vector at time t and location x, containing the six non redundant terms in the stress tensor.

The derivation of Segalman and Reese [4] shows the probability of the von Mises stress being less than some value Y to be given by

$$P(p < Y) = \int_{E(\{D\}, Y)} \prod \rho_r(y_r) \prod dy_r , \qquad \text{Eq 3}$$

where $E(\{D\}, Y)$ is the N-dimensional ellipsoid containing points of the process y that generate von Mises stress less than Y:

$$E(\{D\}, Y) = \{y : (y^T D^2 y) \le Y\} . \qquad \text{Eq 4}$$

The diagonal matrix D is obtained through a process that involves the covariance matrix of the modal coordinates and the tensor-valued stress distributions associated with each mode. The covariance matrix is obtained in part from the cross spectral density matrix. Considerations of the lineage of D show that it can have dimension at most five. See [4] for a detailed discussion of this derivation.

The derivation referred to above also illustrated that the root-mean-square value of von Mises stress can be expressed in terms of D,

$$\bar{p} = \sqrt{E[p^2]} = \sqrt{\sum_r D_r^2} . \qquad \text{Eq 5}$$

In general, the integral of Equation 3 cannot be evaluated analytically. A method for numerical quadrature that yields upper and lower bounds for the that integral was introduced by Segalman and Reese [4]. That quadrature is sufficiently efficient that examining the probability distribution of von Mises stress at selected locations is now tractable.

3. Example Problem

Here we apply the analytical methods discussed above to the hollow cylindrical structure shown in Fig. 1. The system has free-free boundary conditions, with a large mass tied at one end. As discussed in [3], transfer functions relating output stress due to input force can be calculated. With these quantities known, the output von Mises stress due to a random load can be computed.

Three random input loads were considered and independently applied to the cylinder tip as shown. Figure 2 illustrates the power spectra of the applied input. These three load cases were

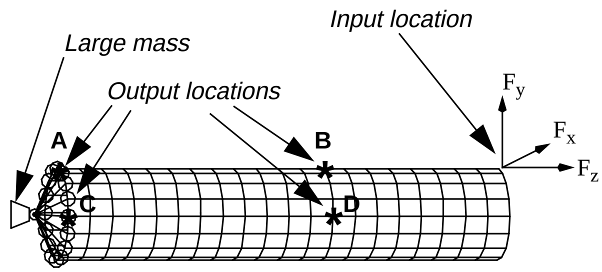

Figure 1. Finite element model of hollow cylinder.

selected to excite the vibrational modes of the system deferentially. In each case, the resulting von Mises probability distribution is evaluated at four points, denoted as *A*, *B*, *C*, and *D* in the figure. These points were selected to show the widest variety in the form of probability distribution of RMS von Mises stress from all excited modes throughout the structure, and for all frequencies of interest.

4. Verification of the analytical method

To check the validity of our analytic methods, time series data was synthesized and compared with the analytical predictions of von Mises stress distributions at each of the four points on the structure. The necessary ingredients of the calculations indicated in [4] were assembled in MATLAB using modal data and transfer functions obtained from MSC/NASTRAN. The resulting $\{D_r\}$ were used in a C language code to calculate the probability distributions.

Time series data was synthesized in the following manner:

- A Fourier expansion for the imposed load was created from the PSD of Fig. 2.
- Random phase information was assigned to the input.
- The above synthesized Fourier expansion for load was multiplied by the appropriate vector-valued stress transfer functions to obtain the Fourier expansions for stress.

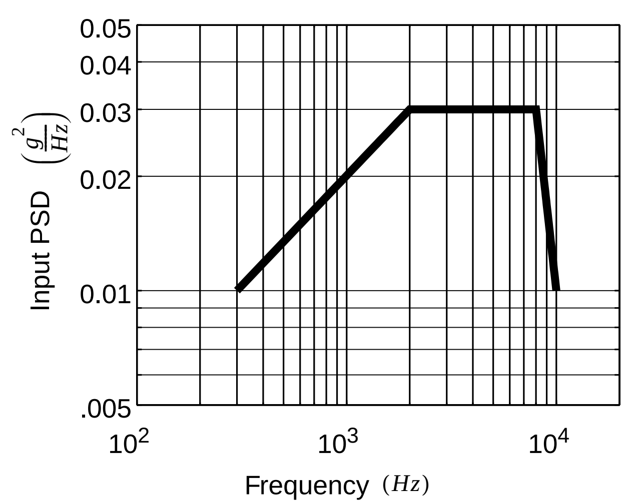

Figure 2. Acceleration PSD imposed at base of cylinder.

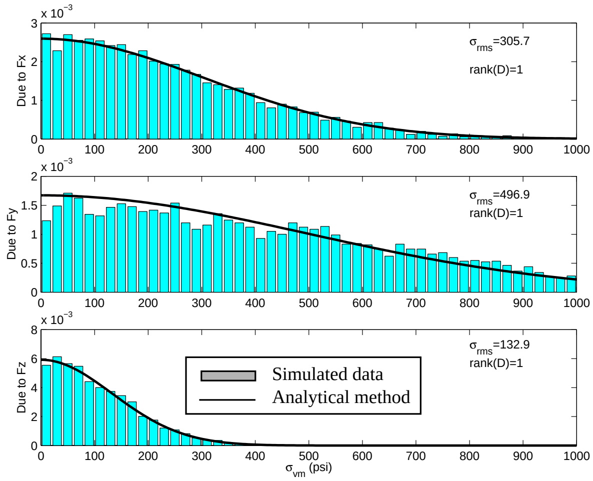

Figure 3. Probability density functions for von Mises stress at location A.

- Time series for stress were obtained by inverse Fourier transform of the above series.
- The quadratic indicated in Equation 1 is evaluated at each time to yield a time series of von Mises stress.
- Histograms are compiled from the above time series.

Because Fourier transforms of very long series must be assembled at each sensor location for each test, the above process is very tedious and not of much practical use. Still it provides a reasonable basis for comparison to the innovative method presented in [4]. The cumulative probability distribution is constrained to increase monotonically from zero to one. As a result, it is difficult to compare details of such curves. Instead, we compare the probability density functions (derivatives of the cumulative probability).

Figure 3 shows the probability density curves of von Mises stress for all three loadings at location A. Shown are both the histograms obtained from the synthetic data and corresponding curves obtained by differentiation of the computed cumulative probability. Very good agreement is achieved. This point was selected for study because it was anticipated that for each loading, all excited vibration modes would contribute to the same stress component. A result of this co-linearity of the stresses is that the rank of matrix D, a quantity indicative of the number of independent random processes experienced at the point of observation, is 1. The derivation in Reference 4 indicates that the form of the probability density for such problems would be the right-hand side of a normal distribution, which agrees with the shape of the curves in Fig. 3.

Figure 4 shows the probability density curves of von Mises stress for all three loadings at location B. The rank of D is one for loadings in the X and Z directions. For those loadings, the distribution is similar in form to the distributions in Fig. 3. The rank of D associated with loading in the Y direction is 2 and the form of the probability density is different: the density is zero at the origin, increases linearly, reaches a maximum and then declines. Again, agreement between the analytic approximation

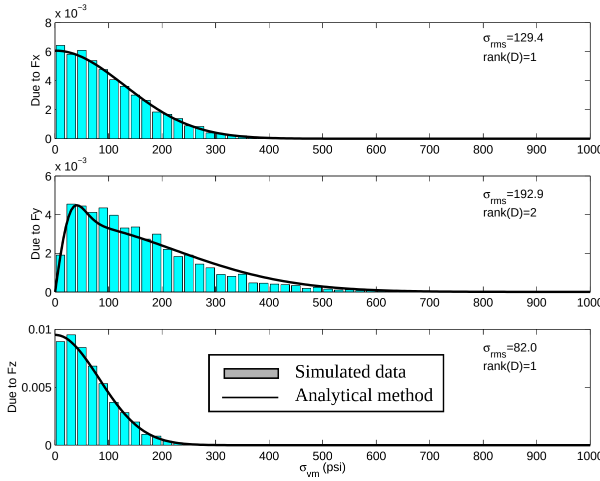

Figure 4. Probability density functions for von Mises stress at location B.

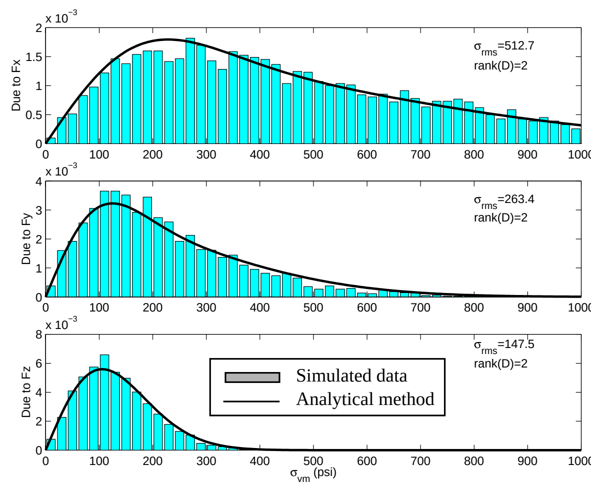

Figure 5. Probability density functions for von Mises stress at location C.

and the histogram of the artificial data is very good.

Figure 5 illustrates the probability density curves of von Mises stress for all three loadings at location C. The rank of D is two for all three load cases. The distributions are similar in form to the distribution associated with the Y loading in Fig. 4. Agreement between the analytic approximation and the histogram of the artificial data is still quite good.

Probability density functions of von Mises stress for the loadings at location D are shown in Fig. 6. The rank of D is two for loadings in the X and Z directions and curves of the form described in the above paragraph are seen again. For the case of loading in the Y direction, the rank of D is three. In this case the form of the analytic probability density is subtly different from what is shown in the other two curves. For this loading, the density is zero at the origin, increases quadratically, reaches a maximum and then declines. Again, we have very good

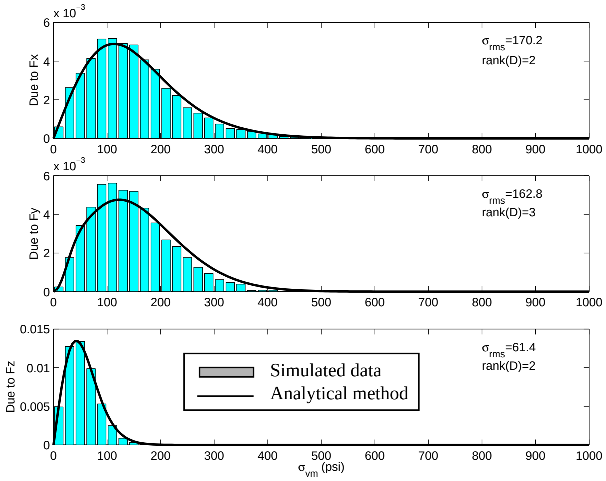

Figure 6. Probability density functions for von Mises stress at location D.

agreement between the histogram of the synthetic data and the analytic approximation.

There is some complexity to the structural response of the cylinder to the applied loads. For the case of loadings in the X and Y directions, the rank of the covariance matrix of the modal coordinates is eight. For loading in the Z direction, the rank is nine. The covariance matrix plays a role in the calculation of D, and its rank is an upper bound for the rank of D. In general the maximum possible rank of D is five, but in problems such as this where there are only three non-redundant components of the stress tensor, the maximum rank of D is three. This restriction is illustrated in Fig. 7, where the rank of D is contoured over the surface of the cylinder.

5. On Reliability Calculations

In dealing with Gaussian distributions, we are in the habit of

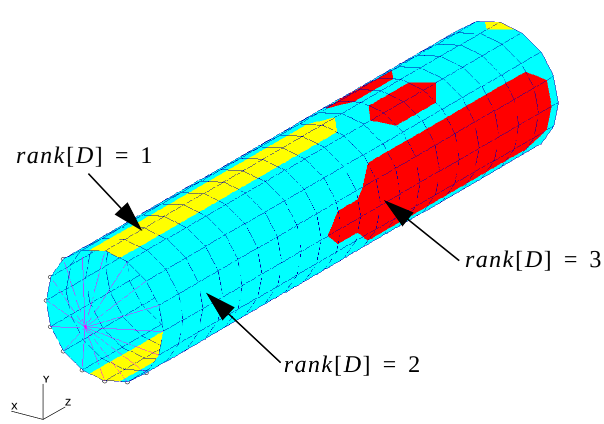

Figure 7. Contours of the rank of D over the surface of the cylinder.

Case No	Rank of D	D_1	D_2	D_3	p such that 0.998<P(p)
1	1	1.0			3.0 $\bar{p}$
2	2	1.0	0.1		3.0 $\bar{p}$
3	3	1.0	1.0	1.0	2.2 $\bar{p}$

Figure 8. Example values of D and comparison with the "three sigma" rule.

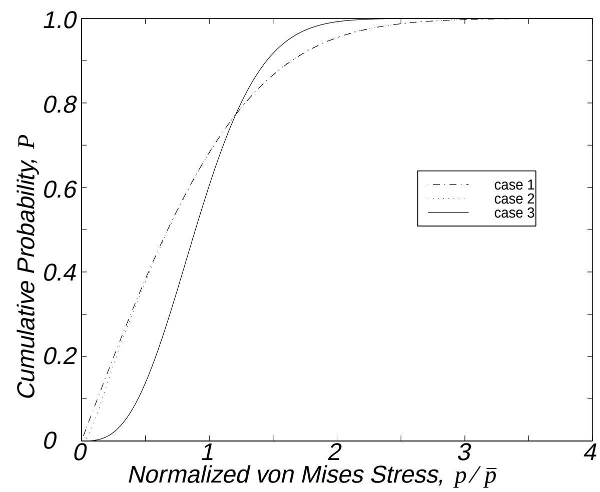

Figure 9. Cumulative probability for three cases of the matrix D.

making assumptions such as that 99.8% of all values will be less than three times the RMS value. Such assumptions are not always appropriate. To illustrate, consider three different cases of the matrix D. These cases are enumerated in Table 1, where the D_i's are the singular values of D.

The cumulative distributions for these three cases are presented in Fig. 9. In this figure, the abscissas are normalized by the RMS value of von Mises stress, $\bar{p}$. For Case 1, having only one random process in effect, the value of von Mises stress below which 99.8% of occurrences lie is $3.0\bar{p}$ - as one expects. For case 2, the three sigma practice again appears adequate. This is not surprising since the D_i's are so similar. For Case 3, on the other hand, the value of von Mises stress below which 99.8% of occurrences lie is $2.2\bar{p}$, showing the "three sigma" rule to be conservative.

Having shown that the "three sigma" rule is not always accurate for a von Mises failure criterion, we now show that the method illustrated in this paper for calculating cumulative probability distribution can be used constructively to make the necessary estimates.

A good representation of the status of a design with respect to a random loading is obtained from Figs. 10 and 11. In Fig. 10, the RMS values of von Mises stress are plotted for the case of loads in the Y direction imposed on the finite element model. Figure 11 shows the contours of the probability that von Mises stress is greater than 2000 psi for the same problem. Note that

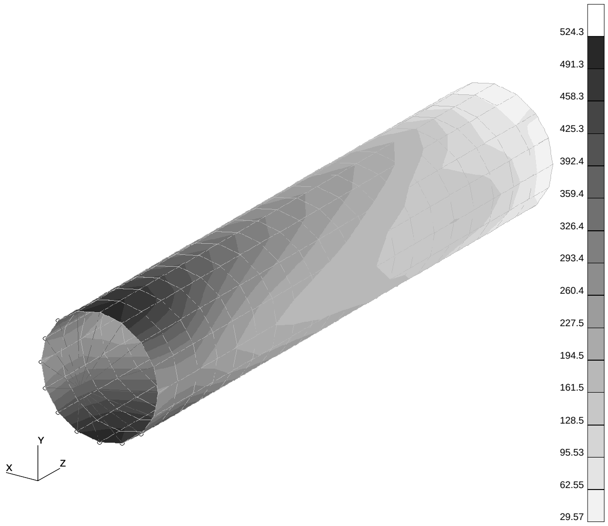

Figure 10. Contours of RMS von Mises stress (psi) resulting from random forces applied in the Y direction at the cylinder

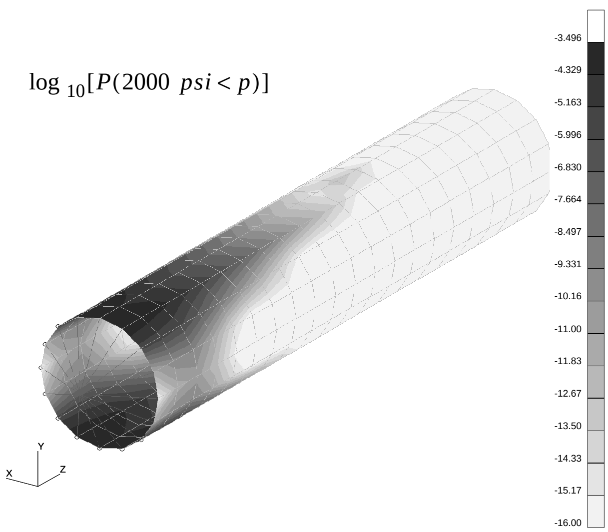

Figure 11. Contour plot of the logarithm of the probability that von Mises stress is greater than 2000 psi.

$P(2000 \text{ psi} < p) = 1 - P(p < 2000 \text{ psi})$. These two pictures provide the designer the information necessary to identify regions that might require redesign.

6. Summary

The method of [4] for calculating the probability distribution for von Mises stress has been illustrated on a simple structure. It has been shown that the number of independent random stress processes taking place varies from place to place on the structure. It is this number of independent processes that determines the form of the probability distribution of von Mises stress at a given point. As a result, one can know if the “three sigma” rule is overly conservative only if the distribution is known at every point. Where reliability with respect to von Mises stress is a design constraint, the method for calculating the probability distribution of von Mises stress introduced in [4] is a tractable and appropriate tool.

References

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